THE ATIYAH CONJECTURE FOR THE HECKE ALGEBRA OF THE INFINITE DIHEDRAL GROUP

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Abstract. We prove a generalized version of the Strong Atiyah Conjecture for the infinite dihedral group W, replacing the group von Neumann algebra $\mathcal{N}W$ with the Hecke-von Neumann algebra $\mathcal{N}_{\mathbf{q}}W$.

1. Introduction

Let W be a discrete group, let $\mathbb{R}W$ denote its group algebra over $\mathbb{R}$, and let L^2W denote the Hilbert space completion of $\mathbb{R}W$ with respect to the standard inner product. Let $\mathcal{N}W$ be the von Neumann algebra obtained by taking the bounded operators on L^2W that commute with the right $\mathbb{R}W$-action. We regarded $\mathcal{N}W$ as an algebra of (left) operators on L^2W. Then any closed $\mathbb{R}W$-invariant subspace $V \subseteq (L^2W)^n$ has a well-defined von Neumann dimension, which we denote by $\dim_W V$. Examples of such subspaces arise naturally in L^2-homology calculations as kernels and image closures of equivariant boundary maps and laplacians, all of which can be represented as right-multiplication by a matrix with entries in $\mathbb{R}W$. The Atiyah Conjecture asserts that any invariant subspace of the form $\ker R_M$ where $R_M : (L^2W)^n \to (L^2W)^m$ is right multiplication by a matrix M with entries in $\mathbb{R}W$ will have rational von Neumann dimension. In full generality this conjecture is false; a counterexample was first given by Austin [1], see also [6, 10]. In all of these counterexamples the group has finite subgroups of arbitrarily large order. For groups with bounded torsion, a stronger form of the conjecture, which specifies denominators of these rational dimensions, is still open. Namely, if Λ denotes the additive subgroup of $\mathbb{R}$ generated by $\{1/|H|\}$ where H ranges over finite subgroups of W, then the Strong Atiyah Conjecture asserts that $\dim_W \ker R_M \in \Lambda$.

In the case where W is a right-angled Coxeter group W, the Strong Atiyah Conjecture was recently settled by Linnell, Okun and Schick [7]. Here we consider a version of Atiyah's question for right-angled Hecke algebras. More precisely, let W be a right-angled Coxeter group with standard generating set S, and let $\mathbb{R}_{\mathbf{q}}W$ denote the Hecke algebra corresponding to W with real deformation multiparameter $\mathbf{q} = (q_s)_{s\in S}$. This algebra has a canonical $\mathbb{R}$-basis $\{T_w \mid w \in W\}$, and multiplication determined by

$$T_sT_w = \begin{cases} T_{sw} & \text{if } |sw| > |w| \\ (q_s - 1)T_w + q_sT_{sw} & \text{if } |sw| < |w| \end{cases}$$

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for all $s \in S$, and $w \in W$. We let q^w denote the product $q_{s_1} \cdots q_{s_n}$ where $s_1 \cdots s_n$ is a reduced expression for w. It follows from Tits' solution to the word problem for W that q^w is independent of the choice of reduced expression. The algebra $\mathbb{R}_{\mathbf{q}}W$ can be regarded as a deformation of the group algebra $\mathbb{R}W$, and the canonical inner product on $\mathbb{R}W$ deforms to the inner product on $\mathbb{R}_{\mathbf{q}}W$ defined by $\langle T_w, T_{w'} \rangle = q^w \delta_{w,w'}$ for all $w, w' \in W$. In particular, the basis elements T_w are orthogonal, and left and right multiplication by T_s (for $s \in S$) are self-adjoint operators. We let $L^2_{\mathbf{q}}W$ denote the Hilbert space completion with respect to this inner product. Again one obtains a von Neumann algebra, which we denote by $\mathcal{N}_{\mathbf{q}}W$, by taking the bounded operators on $L^2_{\mathbf{q}}W$ that commute with the right $\mathbb{R}_{\mathbf{q}}W$-action. And again one obtains von Neumann dimensions for closed $\mathbb{R}_{\mathbf{q}}W$-invariant subspaces $V \subseteq (L^2_{\mathbf{q}}W)^n$. We denote this dimension by $\dim^{\mathbf{q}}_W V$. Translating Atiyah's question to this setting, we then have.

Question. *Let M be an $n \times m$ matrix with entries in $\mathbb{R}_{\mathbf{q}}W$, and let $\ker R_M \subseteq (L^2_{\mathbf{q}}W)^n$ be the kernel of right-multiplication by M. What are the possible values for $\dim^{\mathbf{q}}_W \ker R_M$?*

The point of this paper is to answer this question for the first nontrivial example, namely, when W is the infinite dihedral group. Although the result is admittedly limited in scope, the proof is surprisingly subtle and much more involved than the corresponding result in the Coxeter group setting. In what follows, we assume W is the infinite dihedral group with generators s and t, and we let G be the infinite cyclic subgroup of index 2 generated by st. The proof of the strong Atiyah Conjecture for W boils down to two facts. First, if $V \subseteq (L^2W)^n$ is a left $\mathbb{R}W$-invariant closed subspace then $\dim_G V = 2 \dim_W V$. This follows from the orthogonal decomposition

$$L^2W = L^2G \oplus (L^2G)s \cong (L^2G)^2.$$

And second, (right) multiplication in L^2G by a nonzero element of the group algebra $\mathbb{R}G$ has trivial kernel. This follows from a Fourier series argument. When $q_s \neq 1$ or $q_t \neq 1$, the argument breaks down in two places: first, $L^2_{\mathbf{q}}G$ and $(L^2_{\mathbf{q}}G)s$ are not orthogonal, and second, $L^2_{\mathbf{q}}G$ has nontrivial submodules of the form $\ker R_M$. We address these difficulties by describing a finer orthogonal decomposition of $L^2_{\mathbf{q}}W$. We then prove the following theorem.

Theorem. *Let W be the infinite dihedral group $\langle s, t \mid s^2 = t^2 = 1 \rangle$, and let $\Lambda_{\mathbf{q}}$ be the additive subgroup of $\mathbb{R}$ generated by 1 and $1/(1+q_s)$ and $1/(1+q_t)$. Then for any finite matrix M with entries in $\mathbb{R}_{\mathbf{q}}W$, one has*

$$\dim^{\mathbf{q}}_W \ker R_M \in \Lambda_{\mathbf{q}}.$$

For right-angled Coxeter groups, there is a canonical isomorphism between $\mathbb{R}_{\mathbf{q}}W$ and the ordinary group algebra $\mathbb{R}W$ (see [9] and Section 2, below). Thus, for any matrix M with entries in $\mathbb{R}W$, there is a corresponding matrix $M_{\mathbf{q}}$ with entries in $\mathbb{R}_{\mathbf{q}}W$, and vice versa. It therefore makes sense to ask about $\dim^{\mathbf{q}}_W \ker R_{M_{\mathbf{q}}}$ *as a function of* $\mathbf{q}$. Our proof of the theorem above actually implies that this function is a continuous, piecewise rational function of $\mathbf{q}$ (see Corollary 5.9, below).

To make the paper easier to follow, we outline here the key steps in the proof of the main theorem. The first step is to identify $\mathbb{R}_qW$ with $\mathbb{R}W$ using the canonical isomorphism and then to pass to the subalgebra $\mathbb{R}G$ where G is the free abelian subgroup

of W generated by the translation st. The advantage of $\mathbb{R}G$ over $\mathbb{R}W$ is that the former is isomorphic to the commutative ring of Laurent polynomials, and matrices over this ring are easier to work with. We then consider the action of the group generator st on $L^2_{\mathbf{q}}G$, letting K_+ and K_- denote the $+1$ and -1-eigenspaces, respectively. We obtain an orthogonal decomposition

$$L^2_{\mathbf{q}}G = K_+ \oplus K_- \oplus K_\emptyset$$

where $K_\emptyset$ is the orthogonal complement of K_+ and K_-. We then show that right multiplication by any element $y \in \mathbb{R}G$, restricted to any of these three summands, is either an isomorphism or the zero map (Proposition 5.1). This follows from two facts. First, being a Laurent poynomial in one variable, y factors into linear factors over $\mathbb{C}$. Second, $+1$ and -1 are the only complex eigenvalues for the action of st on $L^2_{\mathbf{q}}G$. Section 3 is devoted entirely to this second fact, which is the main technical result of the paper.

We then extend this decomposition to $L^2_{\mathbf{q}}W$, proving that

$$L^2_{\mathbf{q}}W = K_+ \oplus K_- \oplus K_\emptyset \oplus K_\emptyset s \tag{1.1}$$

as $\mathcal{N}_{\mathbf{q}}G$-modules (Proposition 4.13).

Remark. For any Coxeter group W, Davis et al. [3, Theorem 9.11] prove a decomposition theorem for $L^2_{\mathbf{q}}W$ that generalizes the decomposition of Solomon [11] for finite Coxeter groups (and the ordinary group algebra). In the case of the infinite dihedral group, the two subspaces K_+ and K_- in our decomposition are not just $\mathcal{N}_{\mathbf{q}}G$-modules, but they are also $\mathcal{N}_{\mathbf{q}}W$-modules, and can be used to give an even finer decomposition of $L^2_{\mathbf{q}}W$ than that in [3]. The subspace K_+ corresponds to either the constant functions or "harmonic" functions (denoted by A^S or H^S, respectively, in [3]), but the invariant subspace K_- is new. It can be regarded as the image of K_+ under one of the "partial j" automorphisms described in [9, Section 9] and is a proper invariant subspace of one of the summands in the decomposition of Davis et al.

Given an $\mathbb{R}W$-invariant subspace $V \subseteq (L^2_{\mathbf{q}}W)^n$, we obtain a corresponding decomposition

$$V = V_+ \oplus V_- \oplus V_\emptyset$$

where $V_+ \subseteq K_+^n$, $V_- \subseteq K_-^n$, and $V_\emptyset \subseteq (K_\emptyset \oplus K_\emptyset s)^n$ (Proposition 4.15). We then prove that if V is the kernel of an $\mathbb{R}W$-matrix, then as $\mathcal{N}_{\mathbf{q}}G$-modules we have isomorphisms,

$$V_+ \cong K_+^a, \;\; V_- \cong K_-^b, \;\; V_\emptyset \cong K_\emptyset^c$$

where a, b, c are nonnegative integers (Lemmas 5.2 and 5.4). The proof of this requires one to first show that right multiplication by an $\mathbb{R}W$-matrix corresponds to right multiplication by an $\mathbb{R}G$-matrix with respect to the decomposition (1.1) above, and then to use the fact that matrices over Laurent polynomial rings are essentially diagonalizable. This means that right multiplication by an $\mathbb{R}G$-matrix on any of the subspaces K_+^n, K_-^n, or $(K_\emptyset \oplus K_\emptyset s)^n \cong K_\emptyset^{2n}$ reduces to the 1-dimensional case, where (by Proposition 5.1, mentioned above), the kernel is either trivial or the entire space.

Finally, we calculate the $\mathcal{N}_{\mathbf{q}}G$-dimensions of the modules $V_+ \cong K_+^a$, $V_- \cong K_-^b$, and $V_\emptyset \cong K_\emptyset^c$ (Lemma 4.9), relate these to their $\mathcal{N}_{\mathbf{q}}W$-dimensions (Lemma 4.16), and then complete the proof (Theorem 5.6).

2. Hecke-von Neumann algebras for Right-angled Coxeter groups

Let W be a right-angled Coxeter group with generating set S, and let $\mathbf{q} = (q_s)_{s\in S}$ be a real-valued S-tuple satisfying $q_s > 0$ for all $s \in S$. We let $\mathbb{R}_{\mathbf{q}}W$ denote the corresponding Hecke algebra and note that in addition to the multiplication formulas from the introduction

$$T_sT_w = \begin{cases} T_{sw} & \text{if } |sw| > |w| \\ (q_s-1)T_w + q_sT_{sw} & \text{if } |sw| < |w| \end{cases},$$

there are analogous right-multiplication formulas

$$T_wT_s = \begin{cases} T_{ws} & \text{if } |ws| > |w| \\ (q_s-1)T_w + q_sT_{ws} & \text{if } |ws| < |w| \end{cases}.$$

In a previous paper, the authors noted that for right-angled Coxeter groups, there is a canonical isomorphism $\phi : \mathbb{R}W \to \mathbb{R}_{\mathbf{q}}W$ of $\mathbb{R}$-algebras induced by

$$\phi(s) = \frac{1-q_s}{1+q_s} + \frac{2}{1+q_s}T_s$$

for all $s \in S$ (see [9][Corollary 9.7]). The Hecke algebra $\mathbb{R}_{\mathbf{q}}W$ has an $\mathbb{R}$-basis $\{T_w\}$ canonically indexed by elements of W: each T_w is a product $T_w = T_{s_1}\cdots T_{s_n}$ where $s_1\cdots s_n$ is a reduced expression for w. We let $\tau_w = \phi^{-1}(T_w)$, keeping in mind that τ_w depends on the choice of $\mathbf{q}$. We then have two bases $\{w \mid w \in W\}$ and $\{\tau_w \mid w \in W\}$ for the group algebra $\mathbb{R}W$ (which coincide if and only if $q_s = 1$ for all $s \in S$). Throughout the paper, we shall denote the unit element $\tau_1 = \phi^{-1}(T_1)$ by 1 and identify $\mathbb{R}$ with the constants $\mathbb{R}\tau_1 \subseteq \mathbb{R}W$. From the definition of ϕ we have, for all $s \in S$,

$$s = \frac{1-q_s}{1+q_s} + \frac{2}{1+q_s}\tau_s, \tag{2.1}$$

and since ϕ is an algebra isomorphism, the multiplication formulas for the Hecke basis T_w correspond to the same formulas for the τ_w basis in the group algebra, namely

$$\tau_s\tau_w = \begin{cases} \tau_{sw} & \text{if } |sw| > |w| \\ (q_s-1)\tau_w + q_s\tau_{sw} & \text{if } |sw| < |w| \end{cases} \tag{2.2}$$

and

$$\tau_w\tau_s = \begin{cases} \tau_{ws} & \text{if } |ws| > |w| \\ (q_s-1)\tau_w + q_s\tau_{ws} & \text{if } |ws| < |w| \end{cases}. \tag{2.3}$$

Pulling back the inner product on $\mathbb{R}_{\mathbf{q}}W$ from the introduction, we obtain, a corresponding inner product $\langle , \rangle_{\mathbf{q}}$ on the group algebra $\mathbb{R}W$. This inner product is given by

$$\langle \tau_w, \tau_{w'} \rangle_{\mathbf{q}} = \langle T_w, T_{w'} \rangle = q^w \delta_{w,w'}$$

for all $w, w' \in W$.

We then identify the Hilbert space completion $L^2_{\mathbf{q}}W$ with the completion of the group algebra $\mathbb{R}W$ with respect to the inner product $\langle , \rangle_{\mathbf{q}}$. As in [2, Section 19.2], one obtains a

von Neumann algebra $\mathcal{N}_{\mathbf{q}}W$ of (left) operators on $L^2_{\mathbf{q}}W$ by taking all bounded operators that commute with the right $\mathbb{R}W$-action. Alternatively, we say that an element $x \in L^2_{\mathbf{q}}W$ is *bounded* if there is some constant C such that $\|xy\| \leq C\|y\|$ for all $y \in \mathbb{R}W$. The von Neumann algebra $\mathcal{N}_{\mathbf{q}}W$ can then be identified with the weak closure of the subset of $L^2_{\mathbf{q}}W$ consisting of bounded elements acting on the left of $\mathbb{R}W$. (Similarly, there is a von Neumann algebra of right operators on $L^2_{\mathbf{q}}W$, which we also denote by $\mathcal{N}_{\mathbf{q}}W$. The context will usually determine which algebra we are using.)

A basic fact we shall need about the inner product $\langle , \rangle_{\mathbf{q}}$ on $L^2_{\mathbf{q}}W$ is that for any generator $s \in S$, left and right multiplication by s and τ_s are self-adjoint.

Proposition 2.4. *For any $s \in S$ and $x, y \in L^2_{\mathbf{q}}W$,*

$$\langle sx, y\rangle_{\mathbf{q}} = \langle x, sy\rangle_{\mathbf{q}} \ \text{ and } \ \langle xs, y\rangle_{\mathbf{q}} = \langle x, ys\rangle_{\mathbf{q}}$$

and

$$\langle \tau_s x, y\rangle_{\mathbf{q}} = \langle x, \tau_s y\rangle_{\mathbf{q}} \ \text{ and } \ \langle x\tau_s, y\rangle_{\mathbf{q}} = \langle x, y\tau_s\rangle_{\mathbf{q}}.$$

Proof. In [5, Proposition 2.1], any Hecke algebra $\mathbb{R}_{\mathbf{q}}W$, together with the involution $*$ defined by $T_w^* = T_{w^{-1}}$ and the inner product defined by $\langle T_w, T_{w'}\rangle = q^w \delta_{w,w'}$, is shown to satisfy the axioms for a Hilbert algebra structure in the sense of Dixmier [4]. In particular, for any $x \in \mathbb{R}_{\mathbf{q}}W$, left (respectively, right) multiplication by x^* is the adjoint of left (resp., right) multiplication by x with respect to $\langle , \rangle$. When W is right-angled, the isomorphism $\phi^{-1} : \mathbb{R}_{\mathbf{q}}W \to \mathbb{R}W$ induces a Hilbert algebra structure on $\mathbb{R}W$ where the inner product is $\langle , \rangle_{\mathbf{q}}$ and the $*$-involution is given by $w^* = w^{-1}$ on the $\{w\}$ basis and $\tau_w^* = \tau_{w^{-1}}$ on the $\{\tau_w\}$ basis. Thus, $s^* = s$, and $\tau_s^* = \tau_s$ for all $s \in S$. □

For any positive integer n, we let $(L^2_{\mathbf{q}}W)^n$ denote the Hilbert space direct sum of n copies of $L^2_{\mathbf{q}}W$, and we let $\epsilon_1, \dots, \epsilon_n$ denote the standard basis; in other words $\epsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 in the ith position represents the element $1 \in \mathbb{R}W$. Any closed (left) $\mathbb{R}W$-invariant subspace $V \subseteq (L^2_{\mathbf{q}}W)^n$ will be called a *Hilbert $\mathcal{N}_{\mathbf{q}}W$-module*, and has *von Neumann dimension* defined by

$$\dim_W^{\mathbf{q}} V = \sum_{i=1}^{n} \langle \mathrm{pr}_V(\epsilon_i), \epsilon_i\rangle_{\mathbf{q}}$$

where $\mathrm{pr}_V : (L^2_{\mathbf{q}}W)^n \to V$ is orthogonal projection onto V. An *isomorphism* of Hilbert modules is an $\mathbb{R}W$-equivariant Hilbert space isomorphism. Isomorphic Hilbert modules have the same von Neumann dimension (see e.g., [8, Theorem 1.12]). Similarly, if G is any subgroup of W, we can restrict the inner product $\langle , \rangle_{\mathbf{q}}$ to $\mathbb{R}G$. The Hilbert space completion $L^2_{\mathbf{q}}G$ can then be identified with the closure of $\mathbb{R}G$ in $L^2_{\mathbf{q}}W$. As above, one defines the von Neumann algebra $\mathcal{N}_{\mathbf{q}}G$ to be the algebra of bounded operators on $L^2_{\mathbf{q}}G$ that commute with the right $\mathbb{R}G$-action. A Hilbert $\mathcal{N}_{\mathbf{q}}G$-module V is defined by replacing W with G in the previous paragraph, and its von Neumann dimension will be denoted by $\dim_G^{\mathbf{q}} V$.

3. The G-action on $L^2_{\mathbf{q}}W$

For the remainder of the paper W will be the infinite dihedral group with standard generators s and t. We let G be the infinite cyclic subgroup generated by the product st, and we consider the operator on $L^2_{\mathbf{q}}W$ defined by right multiplication by st. We shall prove that the only possible eigenvalues for this operator are 1 and -1 (and even these may or may not occur depending on the values of the parameters q_s and q_t). The same result holds for left multiplication by st, as well, with the same resulting eigenvalues and eigenvectors, but we shall omit the argument since it is virtually identical to that for right-multiplication.

We work both with the orthogonal basis $\{\tau_w\}$ for $L^2_{\mathbf{q}}W$ and the orthonormal basis $\{\widetilde{\tau}_w\}$ defined by

$$\widetilde{\tau}_w = q^{-w/2}\tau_w,$$

where $q^{-w/2}$ denotes the (positive) real number $1/\sqrt{q^w}$. For the $\mathbb{R}W$-action on $L^2_{\mathbf{q}}W$, we introduce the special elements a_s and a_t defined by

$$a_s := \frac{1+s}{2} = \frac{1+\tau_s}{1+q_s} \quad\text{and}\quad a_t := \frac{1+t}{2} = \frac{1+\tau_t}{1+q_t} \tag{3.1}$$

(the equations follow from (2.1)).

One checks easily using the fact that $s^2 = 1$ and $t^2 = 1$ that a_s and a_t are self-adjoint idempotents, as are their complements $h_s = 1 - a_s$ and $h_t = 1 - a_t$. The latter are given in terms of the bases $\{w\}$ and $\{\tau_w\}$ by

$$h_s = \frac{1-s}{2} = \frac{q_s-\tau_s}{1+q_s} \quad\text{and}\quad h_t = \frac{1-t}{2} = \frac{q_t-\tau_t}{1+q_t}. \tag{3.2}$$

Our first step is to replace the operator st with $a_s - a_t$.

Lemma 3.3. *The vector $\nu \in L^2_{\mathbf{q}}W$ is an eigenvector for st with eigenvalue λ if and only if ν is an eigenvector for $a_s - a_t$ with eigenvalue*

$$\mu = \pm\sqrt{\frac{1}{2} - \frac{1}{2}\operatorname{Re}\lambda}.$$

Proof. Let ν be an eigenvector for st with eigenvalue λ. Since s and t are self-adjoint involutions, st is a unitary operator with $(st)^* = ts$. It follows that $|\lambda| = 1$. Moreover, ν will be in the kernel of the operator

$$(st-\lambda)(st-\overline{\lambda}) = (st)^2 + 1 - 2st\operatorname{Re}\lambda = (st + ts - 2\operatorname{Re}\lambda)(st).$$

Since st is invertible, ν will therefore be an eigenvector for $st+ts$ with eigenvalue $2\operatorname{Re}(\lambda)$. Using the definition of a_s and a_t in (3.1), we have $s = 2a_s - 1$ and $t = 2a_t - 1$, hence

$$st + ts = 4(a_sa_t + a_ta_s) - 4(a_s + a_t) + 1 = 1 - 4(a_s - a_t)^2,$$

where the last expression follows from $a_s^2 = a_s$ and $a_t^2 = a_t$. It follows that ν is an eigenvector for $a_s - a_t$ with eigenvalue $\pm\sqrt{\frac{1}{2} - \frac{1}{2}\operatorname{Re}\lambda}$. Tracing the argument backward gives the reverse implication. $\square$

Next we compute the action of $a_s - a_t$ on the basis vectors $\{\tau_w\}$. To avoid denominators, we let $c = (1+q_s)(1+q_t)$ and let R be the operator

$$R = c(a_s - a_t) = (q_t - q_s) + (1+q_t)\tau_s - (1+q_s)\tau_t.$$

Any eigenvector of $a_s - a_t$ with eigenvalue μ will then be a nonzero vector in the kernel of $R - c\mu$. We compute the products $\tau_w(R - c\mu)$ using the formulas for right-multiplication by τ_s and τ_t:

$$\tau_1(R - c\mu) = (q_t - q_s - c\mu)\tau_1 + (1+q_t)\tau_s - (1+q_s)\tau_t$$

and (for $|ws| > |w|$)

$$\begin{aligned}\tau_{ws}(R - c\mu) &= (q_t - q_s - c\mu)\tau_{ws} + (1+q_t)[(q_s-1)\tau_{ws} + q_s\tau_w] - (1+q_s)\tau_{wst} \\ &= -(1+q_s)\tau_{wst} + (q_sq_t - 1 - c\mu)\tau_{ws} + q_s(1+q_t)\tau_w\end{aligned}$$

and (for $|wt| > |w|$)

$$\begin{aligned}\tau_{wt}(R - c\mu) &= (q_t - q_s - c\mu)\tau_{ws} + (1+q_t)\tau_{wts} - (1+q_s)[(q_t-1)\tau_{wt} + q_t\tau_w] \\ &= (1+q_t)\tau_{wts} - (q_sq_t - 1 + c\mu)\tau_{wt} - q_t(1+q_s)\tau_w.\end{aligned}$$

Using the substitutions $\tau_w = q^{w/2}\widetilde{\tau}_w$, we obtain formulas with respect to the orthonormal basis:

$$\widetilde{\tau}_1(R - c\mu) = \sqrt{q_s}(1+q_t)\widetilde{\tau}_s - \sqrt{q_t}(1+q_s)\widetilde{\tau}_t + (q_t - q_s - c\mu)\widetilde{\tau}_1 \tag{3.4}$$

and (for $|ws| > |w|$)

$$\widetilde{\tau}_{ws}(R - c\mu) = -\sqrt{q_t}(1+q_s)\widetilde{\tau}_{wst} + (q_sq_t - 1 - c\mu)\widetilde{\tau}_{ws} + \sqrt{q_s}(1+q_t)\widetilde{\tau}_w \tag{3.5}$$

and (for $|wt| > |w|$)

$$\widetilde{\tau}_{wt}(R - c\mu) = \sqrt{q_s}(1+q_t)\widetilde{\tau}_{wts} - (q_sq_t - 1 + c\mu)\widetilde{\tau}_{wt} - \sqrt{q_t}(1+q_s)\widetilde{\tau}_w. \tag{3.6}$$

Now suppose ν is an eigenvector for st with eigenvalue λ (hence an eigenvector for R with eigenvalue $c\mu$). For each $w \in W$, let $\{x_w\}$ be the coordinates of ν with respect to the orthonormal basis $\{\widetilde{\tau}_w\}$, i.e., $x_w = \langle \nu, \widetilde{\tau}_w\rangle_{\mathbf{q}}$. We then have

$$\nu = \sum_{w \in W} x_w \widetilde{\tau}_w,$$

and $\nu \in L^2_{\mathbf{q}}W$ if and only if $\sum_w |x_w|^2 < \infty$.

Rewriting the equation $\nu(R - c\mu) = 0$ in terms of the coordinates $\{x_w\}$ using (3.4), (3.5), (3.6), we obtain the equations

$$(q_t - q_s - c\mu)x_1 + \sqrt{q_s}(1+q_t)x_s - \sqrt{q_t}(1+q_s)x_t = 0,$$

and (for $|ws| > |w|$)

$$\sqrt{q_s}(1+q_t)x_w + (q_sq_t - 1 - c\mu)x_{ws} - \sqrt{q_t}(1+q_s)x_{wst} = 0,$$

and (for $|wt| > |w|$)

$$-\sqrt{q_t}(1+q_s)x_w - (q_sq_t - 1 + c\mu)x_{wt} + \sqrt{q_s}(1+q_t)x_{wts} = 0.$$

With the substitutions

$$\alpha_s = \sqrt{q_s} + \frac{1}{\sqrt{q_s}} \qquad \delta = \frac{\alpha_s}{\alpha_t}$$

$$\alpha_t = \sqrt{q_t} + \frac{1}{\sqrt{q_t}} \qquad \text{and} \qquad \beta = \frac{\alpha_{st}}{\alpha_s} - \alpha_t \mu \tag{3.7}$$

$$\alpha_{st} = \sqrt{q_s q_t} - \frac{1}{\sqrt{q_s q_t}} \qquad \gamma = \frac{\alpha_{st}}{\alpha_t} + \alpha_s \mu$$

these three equations simplify to

$$\frac{x_s}{\alpha_s} - \frac{x_t}{\alpha_t} = \left(\mu - \frac{1}{1+q_s} + \frac{1}{1+q_t}\right) x_1, \tag{3.8}$$

and (for $|ws| > |w|$)

$$x_{wst} = \delta^{-1} x_w + \beta x_{ws}, \tag{3.9}$$

and (for $|wt| > |w|$)

$$x_{wts} = \delta x_w + \gamma x_{wt}. \tag{3.10}$$

Applying these last two formulas consecutively to x_{wsts} we have

$$x_{wsts} = \gamma\delta^{-1} x_w + (\delta + \beta\gamma) x_{ws}, \tag{3.11}$$

and applying them to x_{wtst}, we have

$$x_{wtst} = \beta\delta x_w + (\delta^{-1} + \beta\gamma) x_{wt}. \tag{3.12}$$

The equations (3.9) and (3.11) give a second order linear recurrence for the coefficients $x_1, x_s, x_{st}, x_{sts}, \ldots$ given in matrix form by

$$\begin{bmatrix} x_{(st)^{n+1}} \\ x_{(st)^{n+1}s} \end{bmatrix} = M \begin{bmatrix} x_{(st)^n} \\ x_{(st)^n s} \end{bmatrix} \qquad \text{where} \qquad M = \begin{bmatrix} \delta^{-1} & \beta \\ \gamma\delta^{-1} & \beta\gamma + \delta \end{bmatrix} \tag{3.13}$$

and the equations (3.10) and (3.12) yield a recurrence for $x_1, x_t, x_{ts}, x_{tst}, \ldots$ given by

$$\begin{bmatrix} x_{(ts)^{n+1}} \\ x_{(ts)^{n+1}t} \end{bmatrix} = N \begin{bmatrix} x_{(ts)^n} \\ x_{(ts)^n t} \end{bmatrix} \qquad \text{where} \qquad N = \begin{bmatrix} \delta & \gamma \\ \beta\delta & \beta\gamma + \delta^{-1} \end{bmatrix} \tag{3.14}$$

for $n = 0, 1, 2, \ldots$. We let $\mathbf{m}$ and $\mathbf{n}$ denote the initial vectors

$$\mathbf{m} = \begin{bmatrix} x_1 \\ x_s \end{bmatrix} \quad \text{and} \quad \mathbf{n} = \begin{bmatrix} x_1 \\ x_t \end{bmatrix}$$

of these recurrences. They are constrained only by the single equation (3.8)

$$\frac{x_s}{\alpha_s} - \frac{x_t}{\alpha_t} = \left(\mu - \frac{1}{1+q_s} + \frac{1}{1+q_t}\right) x_1.$$

Note that the matrices M and N from (3.13) and (3.14) have the same trace and determinant

$$\operatorname{tr} M = \operatorname{tr} N = \beta\gamma + \delta + \delta^{-1} \quad \text{and} \quad \det M = \det N = 1,$$

hence they have the same eigenvalues. Moreover, these eigenvalues are multiplicative inverses of each other. The basic fact we shall use to eliminate most of the possible

eigenvectors for $a_s - a_t$ is that a nonzero solution $\nu = \sum_w x_w \widetilde{\tau}_w$ to the recurrence (3.13) (and similarly for (3.14)) must satisfy $M^n \mathbf{m} \to 0$ as $n \to \infty$. Otherwise, the sum

$$\sum_{n=0}^{\infty} \|M^n \mathbf{m}\|^2 = \sum_{n=0}^{\infty} (|x_{(st)^n}|^2 + |x_{(st)^n s}|^2),$$

which is a lower bound for $\|\nu\|^2 = \sum_w |x_w|^2$, will diverge.

First we rule out the case where M and N do not have a basis of eigenvectors. In particular, M and N will only have one eigenvalue in this case, and it will be equal to $+1$ or -1.

Lemma 3.15. *If M (and hence N) does not have linearly independent eigenvectors and the initial vectors $\mathbf{m}$ and $\mathbf{n}$ are not both zero, then $\sum_w |x_w|^2 = \infty$.*

Proof. Assume without loss of generality that $\mathbf{m}$ is nonzero, and let $\chi \in \{1, -1\}$ be the eigenvalue for M. Since the χ- eigenspace for M is 1-dimensional, the Jordan form for M will be upper triangular with χ on the diagonal and a 1 in the upper corner. It follows that there exists a basis $\{\mathbf{m}_1, \mathbf{m}_2\}$ such that

$$M^n \mathbf{m}_1 = \chi^n \mathbf{m}_1, \text{ and } M^n \mathbf{m}_2 = \chi^n \mathbf{m}_2 + n\chi^{n-1} \mathbf{m}_1.$$

Writing $\mathbf{m} = a\mathbf{m}_1 + b\mathbf{m}_2$, we then have

$$M^n \mathbf{m} = (a\chi + bn)\chi^{n-1} \mathbf{m}_1 + b\chi^n \mathbf{m}_2.$$

Since a and b are not both zero and $\chi = \pm 1$, the sequence $M^n \mathbf{m}$ does not converge to zero. □

Now assume M and N each have linearly independent eigenvectors $\mathbf{m}_1, \mathbf{m}_2$ and $\mathbf{n}_1, \mathbf{n}_2$, respectively. Since M and N have the same eigenvalues, we can assume further that $\mathbf{m}_i$ and $\mathbf{n}_i$ correspond to the same eigenvalue, which we denote by χ_i. Since $\chi_1 \chi_2 = 1$, we also assume $|\chi_1| \geq 1 \geq |\chi_2| > 0$. Our next step is to rule out the case where either of the initial vectors has a nonzero component in the direction of the χ_1-eigenvector.

Lemma 3.16. *Assume the initial vectors $\mathbf{m}$ and $\mathbf{n}$ are expressed as linear combinations of $\{\mathbf{m}_1, \mathbf{m}_2\}$ and $\{\mathbf{n}_1, \mathbf{n}_2\}$, respectively. If $\mathbf{m}$ has a nonzero component in the direction of $\mathbf{m}_1$ or $\mathbf{n}$ has a nonzero component in the direction of $\mathbf{n}_1$ then $\sum_w |x_w|^2 = \infty$.*

Proof. Suppose $\mathbf{m} = a\mathbf{m}_1 + b\mathbf{m}_2$ with $a \neq 0$. Then

$$M^n \mathbf{m} = a(\chi_1)^n \mathbf{m}_1 + b(\chi_2)^n \mathbf{m}_2.$$

Since $|\chi_1| \geq 1$, these vectors do not converge to zero. The $\mathbf{n}$ case is similar. □

In light of Lemmas 3.15 and 3.16, we may assume that if $\nu = \sum_w x_w \widetilde{\tau}_w$ is an eigenvector of $a_s - a_t$ with eigenvalue μ, then

- M (and also N) has distinct eigenvalues χ_1 and χ_2 with $|\chi_1| > 1 > |\chi_2|$, and
- $\mathbf{m}$ (respectively, $\mathbf{n}$) is a χ_2-eigenvector of M (resp., N).

We consider the following two cases.

Case 1. Either $\beta = 0$ and $\chi_2 = \delta^{-1}$ or $\gamma = 0$ and $\chi_2 = \delta$.

Case 2. The vectors

$$\mathbf{m}' = \begin{bmatrix} \beta \\ \chi_2 - \delta^{-1} \end{bmatrix} \text{ and } \mathbf{n}' = \begin{bmatrix} \gamma \\ \chi_2 - \delta \end{bmatrix}$$

are both nonzero.

We first rule out Case 1. Suppose $\beta = 0$ and $\chi_2 = \delta^{-1}$. Since $\beta = 0$, the matrices M and N simplify to

$$M = \begin{bmatrix} \delta^{-1} & 0 \\ \gamma\delta^{-1} & \delta \end{bmatrix} \text{ and } N = \begin{bmatrix} \delta & \gamma \\ 0 & \delta^{-1} \end{bmatrix},$$

and

$$\mu = \frac{\alpha_{st}}{\alpha_s\alpha_t} = \frac{q_sq_t - 1}{(q_s+1)(q_t+1)}.$$

Since $\chi_2 = \delta^{-1}$, a simple calculation then shows that the χ_2-eigenvectors of M and N are

$$\begin{bmatrix} q_s - q_t \\ -2\sqrt{q_s}(1+q_t) \end{bmatrix} \text{ and } \begin{bmatrix} -2\sqrt{q_t}(1+q_s) \\ q_s - q_t \end{bmatrix},$$

respectively. Since q_s and q_t are positive reals, the first coordinates of these vectors cannot both be zero. On the other hand, since these vectors are nonzero multiples of $\mathbf{m}$ and $\mathbf{n}$ (which both have first coordinate equal to x_1), neither of these two vectors can have vanishing first coordinate. It follows that $x_1 \neq 0$, so we can scale ν so that $x_1 = 1$. Then $\mathbf{m} = \begin{bmatrix} 1 \\ x_s \end{bmatrix}$ and $\mathbf{n} = \begin{bmatrix} 1 \\ x_t \end{bmatrix}$. Since these are multiples of the χ_2-eigenvectors above, we have

$$x_s = -\frac{2\sqrt{q_s}(q_t+1)}{q_s - q_t},$$

and

$$x_t = -\frac{q_s - q_t}{2\sqrt{q_t}(1+q_s)}.$$

Substituting these values into the initial equation (3.8), and isolating the numerator, we obtain

$$(q_s + q_t + 2)(2q_sq_t + q_s + q_t) = 0$$

which has no solutions for positive q_s and q_t. A similar analysis yields a contradiction in the case $\gamma = 0$ and $\chi_2 = \delta$.

For Case 2, the vectors $\mathbf{m}'$ and $\mathbf{n}'$ are nonzero. A calculation shows that they are χ_2-eigenvectors for M and N, respectively, hence are nonzero multiples of $\mathbf{m}$ and $\mathbf{n}$. We can assume that β and γ are not both zero. (Otherwise, both M and N would be diagonal with entries δ and δ^{-1}, which means χ_2 would have to be one of these, putting us back into Case 1.) Moreover, since $\mathbf{m}'$ and $\mathbf{n}'$ are nonzero multiples of the vectors $\mathbf{m}$ and $\mathbf{n}$, respectively, and the latter both have the same first coordinate x_1, we know that neither β nor γ can be zero. Again, by scaling ν if necessary to get $x_1 = 1$, we then have

$$x_s = \frac{\chi_2 - \delta^{-1}}{\beta} \tag{3.17}$$

and since $\mathbf{n} = \begin{bmatrix} x_1 \\ x_t \end{bmatrix}$ is a multiple of $\mathbf{n}_2$, we have

$$x_t = \frac{\chi_2 - \delta}{\gamma}. \tag{3.18}$$

Substituting these values into the initial equation (3.8) we obtain

$$\frac{\chi_2 - \delta^{-1}}{\beta\alpha_s} - \frac{\chi_2 - \delta}{\gamma\alpha_t} = \mu - \frac{1}{1+q_s} + \frac{1}{1+q_t}.$$

On the other hand, χ_2 must also satisfy the characteristic equation for M and N, which is

$$\chi_2^2 - (\beta\gamma + \delta + \delta^{-1})\chi_2 + 1 = 0$$

Rewriting these equations in terms of q_s and q_t, and solving simultaneously for χ_2 and μ, we obtain the solutions

- $\chi_2 = \sqrt{q_s q_t}$ and $\mu = 0$,
- $\chi_2 = 1/\sqrt{q_s q_t}$ and $\mu = 0$,
- $\chi_2 = -\sqrt{q_s}/\sqrt{q_t}$ and $\mu = 1$, or
- $\chi_2 = -\sqrt{q_t}/\sqrt{q_s}$ and $\mu = -1$.

It follows that the only possible eigenvalues for $a_s - a_t$ are $\mu = 0$ and $\mu = \pm 1$, and hence (by Lemma 3.3), the only possible eigenvalues for st are $\lambda = +1$ (if $\mu = 0$) and $\lambda = -1$ (if $\mu = \pm 1$).

To describe the corresponding eigenvectors in a concise way, we define for any real parameters r_s, r_t the vector $\kappa(r_s, r_t)$ as follows. For each $w \in W$, we define the coefficient r^w as we did q^w. For the dihedral group, this looks like

$$r^w = \begin{cases} r_s^n r_t^n & \text{if } w = (st)^n \text{ or } w = (ts)^n \\ r_s^{n+1} r_t^n & \text{if } w = (st)^n s \\ r_s^n r_t^{n+1} & \text{if } w = t(st)^n \end{cases} \tag{3.19}$$

for all $n \geq 0$. We then define $\kappa(r_s, r_t)$ by

$$\kappa(r_s, r_t) = \sum_w r^w \tau_w.$$

The L^2-norm of $\kappa(r_s, r_t)$ is given by the geometric series

$$\begin{aligned} \|\kappa(r_s, r_t)\|^2 &= \sum_w (r^w)^2 q^w \\ &= 1 + r_s^2 q_s + r_t^2 q_t + \sum_{n=1}^{\infty} (2 + r_s^2 q_s + r_t^2 q_t)(r_s r_t)^{2n} (q_s q_t)^n. \end{aligned}$$

This series converges if and only if

$$(r_s r_t)^2 < \frac{1}{q_s q_t},$$

and in this case converges to

$$\|\kappa(r_s, r_t)\|^2 = \frac{(1 + r_s^2 q_s)(1 + r_t^2 q_t)}{1 - r_s^2 r_t^2 q_s q_t}. \tag{3.20}$$

Putting all of this together, we obtain the following theorem.

Theorem 3.21. *If λ is an eigenvalue for right or left multiplication by st on $L^2_{\mathbf{q}}W$, then $\lambda \in \{-1, +1\}$ and the corresponding eigenspace is spanned by a single vector. The eigenvalue/eigenvector pairs occur as follows:*

(1) *If $q_s q_t < 1$, then $\lambda = 1$ occurs with eigenvector $\kappa(1, 1)$,*
(2) *If $q_s q_t > 1$, then $\lambda = 1$ occurs with eigenvector $\kappa(-1/q_s, -1/q_t)$,*
(3) *If $q_s < q_t$, then $\lambda = -1$ occurs with eigenvector $\kappa(1, -1/q_t)$,*
(4) *If $q_s > q_t$, then $\lambda = -1$ occurs with eigenvector $\kappa(-1/q_s, 1)$.*

Proof. For right multiplication by st, the only thing left to prove is that the indicated eigenvectors are the solutions to the recurrences (3.13) and (3.14) for the given values of λ and $\mathbf{q}$. Using the initial vectors $\mathbf{m} = \begin{bmatrix} 1 \\ x_s \end{bmatrix}$, $\mathbf{n} = \begin{bmatrix} 1 \\ x_t \end{bmatrix}$ to get

$$\begin{aligned} x_{(st)^n} &= (\chi_2)^n, \\ x_{(ts)^n} &= (\chi_2)^n, \\ x_{(st)^n s} &= (\chi_2)^n x_s, \\ x_{(ts)^n t} &= (\chi_2)^n x_t, \end{aligned}$$

with x_s and x_t given by (3.17) and (3.18). If, for example, $\lambda = 1$ and $q_s q_t < 1$, then $\mu = 0$ and $\chi_2 = \sqrt{q_s q_t}$. It follows that $x_1 = 1$, $x_s = \sqrt{q_s}$, $x_t = \sqrt{q_t}$, and in general $x_w = q^{w/2}$. Hence

$$\nu = \sum_w q^{w/2} \widetilde{\tau}_w = \sum_w \tau_w = \kappa(1, 1),$$

which is in $L^2_{\mathbf{q}}W$. The cases (2)-(4) are similar.

For left multiplication, one notes that ν is a λ-eigenvector for right multiplication by (st) if and only if ν^* is a $\overline{\lambda}$-eigenvector for left multiplication by $(ts) = (st)^*$. But since $|\lambda| = 1$, this is true if and only if ν^* is a λ-eigenvector for left multiplication by $st = (ts)^{-1}$. The result then follows from the fact that for real values of r_s and r_t, $\kappa(r_s, r_t)$ is self-adjoint. $\square$

4. Decompositions of $\mathcal{N}_{\mathbf{q}}G$ and $\mathcal{N}_{\mathbf{q}}W$-modules

In this section we use the eigenspaces for the st-action to obtain orthogonal decompositions of $L^2_{\mathbf{q}}G$ and $L^2_{\mathbf{q}}W$. We then use these decompositions to decompose any $\mathcal{N}_{\mathbf{q}}W$-module in order to relate its von Neumann dimension as an $\mathcal{N}_{\mathbf{q}}G$-module to its dimension as an $\mathcal{N}_{\mathbf{q}}W$-module.

First we describe key properties of the eigenvectors in Theorem 3.21. For a given $\mathbf{q}$, we let κ_+ denote the vector

$$\kappa_+ = \begin{cases} \kappa(1, 1) & \text{if } q_s q_t < 1 \\ \kappa(-1/q_s, -1/q_t) & \text{if } q_s q_t > 1 \\ 0 & \text{if } q_s q_t = 1 \end{cases}$$

and we let κ_- denote the vector

$$\kappa_- = \begin{cases} \kappa(1, -1/q_t) & \text{if } q_s < q_t \\ \kappa(-1/q_s, 1) & \text{if } q_s > q_t \\ 0 & \text{if } q_s = q_t \end{cases} .$$

Remark 4.1. Many of the results of this section follow from results of Davis et al. [3]. In particular, for $q_s q_t < 1$ the span of κ_+ is the invariant subspace of $L^2_{\mathbf{q}}W$ consisting of constants, which is denoted by $A^{\{s,t\}}$ in [3]. Projection onto this subspace is the averaging operator denoted by $a_{\{s,t\}}$ in [3] and by $\widetilde{\kappa}_+$, below. The vectors $\kappa_\pm$ for other values of $\mathbf{q}$ can all be obtained from κ_+ by applying the "partial j-automorphisms" of $L^2_{\mathbf{q}}W$ described in [9, Section 9]. For completeness, we present proofs here without using these more general results.

Proposition 4.2. *Any element $w \in W$ fixes the vectors κ_+ and κ_- (up to sign). More precisely, we have:*

1. $s\kappa_+ = \kappa_+ s = \kappa_+$ *and* $t\kappa_+ = \kappa_+ t = \kappa_+$ *if* $q_s q_t < 1$,
2. $s\kappa_+ = \kappa_+ s = -\kappa_+$ *and* $t\kappa_+ = \kappa_+ t = -\kappa_+$ *if* $q_s q_t > 1$,
3. $s\kappa_- = \kappa_- s = \kappa_-$ *and* $t\kappa_- = \kappa_- t = -\kappa_-$ *if* $q_s < q_t$, *and*
4. $s\kappa_- = \kappa_- s = -\kappa_-$ *and* $t\kappa_- = \kappa_- t = \kappa_-$ *if* $q_s > q_t$.

Proof. These are all calculations using Hecke multiplication. The two basic identities one needs are $sa_s = a_s$ and $sh_s = -h_s$. These follows from the definitions of a_s and h_s in (3.1) and (3.2) in terms of the group algebra basis:

$$sa_s = \frac{s(1+s)}{2} = \frac{s+s^2}{2} = \frac{s+1}{2} = a_s,$$

and

$$sh_s = \frac{s(1-s)}{2} = \frac{s-s^2}{2} = \frac{s-1}{2} = -h_s.$$

Rewriting these identities using the expressions for a_s and h_s using the Hecke algebra basis in (3.1) and (3.2), and multiplying both sides by $1 + q_s$, we obtain the identities

$$s(1+\tau_s) = 1 + \tau_s \quad \text{and} \quad s(q_s - \tau_s) = -(q_s - \tau_s). \tag{4.3}$$

Now to get, for example, the identity $s\kappa_+ = \kappa_+$ when $q_s q_t < 1$, we have

$$\begin{aligned} s\kappa_+ &= s\kappa(1,1) \\ &= s(1 + \tau_s + \tau_t + \tau_{st} + \tau_{ts} + \tau_{sts} + \cdots) \\ &= s(1+\tau_s)(1 + \tau_t + \tau_{ts} + \cdots) \\ &= (1+\tau_s)(1 + \tau_t + \tau_{ts} + \cdots) \\ &= \kappa_+. \end{aligned}$$

To get the identity $s\kappa_- = -\kappa_-$ when $q_s > q_t$, we have

$$\begin{aligned} s\kappa_- &= s\kappa(-1/q_s, 1) \\ &= s(1 - \tau_s/q_s + \tau_t - \tau_{st}/q_s - \tau_{ts}/q_s + \tau_{sts}/q_s^2 - \cdots) \\ &= s(q_s - \tau_s)(1/q_s + \tau_t/q_s - \tau_{ts}/q_s^2 - \cdots) \\ &= -(q_s - \tau_s)(1/q_s + \tau_t/q_s + \tau_{ts}/q_s^2 - \cdots) \\ &= -\kappa_-. \end{aligned}$$

The remaining identities are obtained in a similar fashion by factoring $(1+\tau_s)$, $(1+\tau_t)$, $(q_s - \tau_s)$, or $(q_t - \tau_t)$ out of $\kappa_\pm$ on the right or left depending on the case. We leave the details to the reader. □

Solving for τ_s in (3.1) we get the formulas

$$\tau_s = \frac{q_s - 1}{2} + \frac{q_s + 1}{2} s \quad \text{and} \quad \tau_t = \frac{q_t - 1}{2} + \frac{q_t + 1}{2} t.$$

Using Proposition 4.2, we then obtain additional formulas for products $\kappa_\pm$ with the Hecke generators τ_s and τ_t:

$$\begin{aligned} &\tau_s\kappa_+ = \kappa_+\tau_s = q_s\kappa_+ \ \text{ and } \ \tau_t\kappa_+ = \kappa_+\tau_t = q_t\kappa_+ \ \text{ if } \ q_sq_t < 1, \\ &\tau_s\kappa_+ = \kappa_+\tau_s = -\kappa_+ \ \text{ and } \ \tau_t\kappa_+ = \kappa_+\tau_t = -\kappa_+ \ \text{ if } \ q_sq_t > 1, \\ &\tau_s\kappa_- = \kappa_-\tau_s = q_s\kappa_- \ \text{ and } \ \tau_t\kappa_- = \kappa_-\tau_t = -\kappa_- \ \text{ if } \ q_s < q_t, \\ &\tau_s\kappa_- = \kappa_-\tau_s = -\kappa_- \ \text{ and } \ \tau_t\kappa_- = \kappa_-\tau_t = q_t\kappa_- \ \text{ if } \ q_s > q_t. \end{aligned} \tag{4.4}$$

These are useful because they allow us to show that the vectors $\kappa_\pm$ extend to well-defined operators in $\mathcal{N}_{\mathbf{q}}W$.

Proposition 4.5. *The elements κ_+ and κ_- acting on $\mathbb{R}W$ extend to bounded operators in $\mathcal{N}_{\mathbf{q}}W$ (and $\mathcal{N}_{\mathbf{q}}G$).*

Proof. Let κ be either κ_+ or κ_-. Since κ commutes with all elements in $\mathbb{R}W$, it suffices to show that for any $y \in \mathbb{R}W$, we have $\|\kappa y\|_{\mathbf{q}} \le C\|y\|_{\mathbf{q}}$ for some constant C. In fact, we'll show that $C = \|\kappa\|_{\mathbf{q}}^2$ works. By definition, κ is one of the four vectors $\kappa(r_s, r_t)$ where (r_s, r_t) is one of the pairs $(1,1), (-1/q_s, -1/q_t), (1, -1/q_t), (-1/q_s, 1)$; hence,

$$\kappa = \sum r^w \tau_w$$

with r^w given by (3.19). Expressing τ_w as a product of τ_s's and τ_t's, and using the product formulas (4.4), one can verify that

$$\kappa\tau_w = q^w r^w \kappa. \tag{4.6}$$

Letting $y = \sum_w y_w\tau_w$, we then have

$$\kappa y = \sum_w y_w \kappa\tau_w = \sum_w y_w q^w r^w \kappa = \sum_w (y_w q^{w/2})(r^w q^{w/2})\kappa.$$

Taking square norms, we have

$$\begin{aligned}
\|\kappa y\|_{\mathbf{q}}^2 &= |\sum_w (y_w q^{w/2})(r^w q^{w/2})|^2 \|\kappa\|_{\mathbf{q}}^2 \\
&\leq \sum_w |y_w q^{w/2}|^2 \sum_w |r^w q^{w/2}|^2 \|\kappa\|_{\mathbf{q}}^2 \\
&= (\sum_w |y_w|^2 q^w)(\sum_w |r^w|^2 q^w) \|\kappa\|_{\mathbf{q}}^2 \\
&= \|y\|_{\mathbf{q}}^2 \|\kappa\|_{\mathbf{q}}^2 \|\kappa\|_{\mathbf{q}}^2,
\end{aligned}$$

and taking square roots gives $\|\kappa y\|_{\mathbf{q}} \leq \|\kappa\|_{\mathbf{q}}^2 \|y\|_{\mathbf{q}}$. □

Let K_+ and K_- denote the $+1$ and -1-eigenspaces (respectively) for the right st-action on $L^2_{\mathbf{q}}W$. In light of Theorem 3.21, K_+ (respectively, K_-) is spanned by the single vector κ_+ (resp., κ_-).

Proposition 4.7. *The subspace $L^2_{\mathbf{q}}G \subseteq L^2_{\mathbf{q}}W$ is st-invariant (on both sides) and contains both K_+ and K_-. In fact, we have an orthogonal decomposition of $\mathcal{N}_{\mathbf{q}}G$-modules given by*

$$L^2_{\mathbf{q}}G = K_+ \oplus K_- \oplus K_\emptyset$$

where $K_\emptyset$ is the orthogonal complement of $K_+ \oplus K_-$ in $L^2_{\mathbf{q}}G$.

Proof. That $L^2_{\mathbf{q}}G$ is st-invariant is clear, as is the orthogonality of K_+ and K_- (st is a unitary operator so its eigenspaces are orthogonal). It only remains to prove then that $K_\pm \subseteq L^2_{\mathbf{q}}G$. For this, we use the fact that the orthogonal projection π from $L^2_{\mathbf{q}}W$ onto $L^2_{\mathbf{q}}G$ is an $\mathcal{N}_{\mathbf{q}}G$-module map, hence commutes with multiplication by st. It follows that π must map st-eigenspaces to st-eigenspaces (with the same eigenvalue). Since K_+ is spanned by the single vector κ_+ we must have either $\pi(\kappa_+) = \kappa_+$ or $\pi(\kappa_+) = 0$. In other words, κ_+ is either in the subspace $L^2_{\mathbf{q}}G$ or it is orthogonal to it. To be orthogonal to $L^2_{\mathbf{q}}G$, one would have to have $\langle \kappa_+, 1\rangle_{\mathbf{q}} = 0$ since $1 \in \mathbb{R}G \subseteq L^2_{\mathbf{q}}G$. But it follows immediately from the definition of κ_+ that either κ_+ is zero (in which case $K_+ \subseteq L^2_{\mathbf{q}}G$, trivially) or $\langle \kappa_+, 1\rangle_{\mathbf{q}} = 1$. Hence $\kappa_+ \in L^2_{\mathbf{q}}G$ and so $K_+ \subseteq L^2_{\mathbf{q}}G$. The same argument applied to the -1-eigenspace for st shows that $K_- \subseteq L^2_{\mathbf{q}}G$. □

It will be convenient to work with the orthogonal projections onto K_+ and K_-. Since K_+ and K_- are the spans of the single vectors κ_+ and κ_-, the relevant projections are simply given by appropriate scalings. We define $\widetilde{\kappa}_+$ and $\widetilde{\kappa}_-$ by

$$\widetilde{\kappa}_+ = \frac{\kappa_+}{\|\kappa_+\|_{\mathbf{q}}^2} \text{ and } \widetilde{\kappa}_- = \frac{\kappa_-}{\|\kappa_-\|_{\mathbf{q}}^2},$$

and we define $\widetilde{\kappa}_\emptyset$ by

$$\widetilde{\kappa}_\emptyset = 1 - \widetilde{\kappa}_+ - \widetilde{\kappa}_-.$$

Proposition 4.8. *The elements $\widetilde{\kappa}_\pm$ and $\widetilde{\kappa}_\emptyset$ are central self-adjoint idempotents in the von Neumann algebras $\mathcal{N}_{\mathbf{q}}G$ and $\mathcal{N}_{\mathbf{q}}W$. In particular, multiplication on the right or left*

by $\widetilde{\kappa}_{\pm}$ defines orthogonal projection from $L^2_{\mathbf{q}}G$ onto $K_{\pm}$ and multiplication by $\widetilde{\kappa}_{\emptyset}$ defines orthogonal projection from $L^2_{\mathbf{q}}G$ onto $K_{\emptyset}$.

Proof. Since $\widetilde{\kappa}_{\pm}$ are multiples of $\kappa_{\pm}$, by Proposition 4.5 they are elements of $\mathcal{N}_{\mathbf{q}}G$ and $\mathcal{N}_{\mathbf{q}}W$. Since $\widetilde{\kappa}_{\emptyset}$ is a finite linear combination of 1, $\widetilde{\kappa}_{+}$ and $\widetilde{\kappa}_{-}$, it is in $\mathcal{N}_{\mathbf{q}}G$ and $\mathcal{N}_{\mathbf{q}}W$ as well. Since all three of these operators commute with every element of $\mathbb{R}W$ (by Proposition 4.2) and $\mathbb{R}W$ is dense in $L^2_{\mathbf{q}}W$, they are all central. Self-adjointness follows from the explicit formulas for κ_{+} and κ_{-}, in which the coefficient of τ_w is always the same as the coefficient of $\tau_w^* = \tau_{w^{-1}}$. It remains to show that they are all idempotent. If κ denotes κ_{+} or κ_{-}, then we have

$$\kappa = \sum_w r^w \tau_w$$

with r^w given by (3.19), hence by (4.6) we have

$$\kappa^2 = \sum_w r^w \tau_w \kappa = \sum_w (r^w)^2 q^w \kappa = \|k\|^2_{\mathbf{q}} \kappa.$$

Dividing both sides by $\|\kappa\|^2_{\mathbf{q}}$ gives $\widetilde{\kappa}^2 = \widetilde{\kappa}$. The operator $\widetilde{\kappa}_{\emptyset} = 1 - \widetilde{\kappa}_{+} - \widetilde{\kappa}_{1}$ is idempotent because it is the orthogonal projection onto the complement of K_{+} and K_{-}. □

Using these idempotents, we can compute $\mathcal{N}_{\mathbf{q}}G$-dimensions of the various pieces in our decomposition.

Lemma 4.9. *The von Neumann dimensions of the $\mathcal{N}_{\mathbf{q}}G$-modules K_{+}, K_{-}, and $K_{\emptyset}$ are given by*

$$\dim^{\mathbf{q}}_G K_{+} = \frac{|1 - q_s q_t|}{(1+q_s)(1+q_t)}, \qquad \dim^{\mathbf{q}}_G K_{-} = \frac{|q_t - q_s|}{(1+q_s)(1+q_t)},$$

and

$$\dim^{\mathbf{q}}_G K_{\emptyset} = \begin{cases} \dfrac{2q_s}{1+q_s} & \text{if } q_s q_t \le 1 \text{ and } q_s \le q_t, \\ \dfrac{2q_t}{1+q_t} & \text{if } q_s q_t \le 1 \text{ and } q_s \ge q_t, \\ \dfrac{2}{1+q_t} & \text{if } q_s q_t \ge 1 \text{ and } q_s \le q_t, \\ \dfrac{2}{1+q_s} & \text{if } q_s q_t \ge 1 \text{ and } q_s \ge q_t. \end{cases}$$

Proof. By definition of von Neumann dimension and the idempotents $\widetilde{\kappa}_{\pm}$, we have

$$\dim^{\mathbf{q}}_G K_{\pm} = \langle \widetilde{\kappa}_{\pm}, 1 \rangle_{\mathbf{q}} = \frac{1}{\|\kappa_{\pm}\|^2_{\mathbf{q}}} \langle \kappa_{\pm}, 1 \rangle_{\mathbf{q}} = \frac{1}{\|\kappa_{\pm}\|^2_{\mathbf{q}}}.$$

Substituting $(r_s, r_t) = (1,1)$ and $(r_s, r_t) = (-1/q_s, -1/q_t)$ into (3.20) to get $\|\kappa_{+}\|^2_{\mathbf{q}}$, we obtain

$$\dim^{\mathbf{q}}_G K_{+} = \langle \widetilde{\kappa}_{+}, 1 \rangle_{\mathbf{q}} = \frac{|1 - q_s q_t|}{(1+q_s)(1+q_t)}, \tag{4.10}$$

and substituting $(r_s, r_t) = (1, -1/q_t)$ and $(r_s, r_t) = (-1/q_t, 1)$ into (3.20) to get $\|\kappa_-\|^2_{\mathbf{q}}$, we obtain

$$\dim_G^{\mathbf{q}} K_- = \langle \widetilde{\kappa}_-, 1\rangle_{\mathbf{q}} = \frac{|q_t - q_s|}{(1+q_s)(1+q_t)}. \tag{4.11}$$

Since $K_\emptyset$ is the orthogonal complement of K_+ and K_- in $L^2_{\mathbf{q}}G$ and $\dim_G^{\mathbf{q}} L^2_{\mathbf{q}}G = 1$, we have

$$\dim_G^{\mathbf{q}} K_\emptyset = 1 - \frac{|1 - q_s q_t|}{(1+q_s)(1+q_t)} - \frac{|q_t - q_s|}{(1+q_s)(1+q_t)}$$

which simplifies to the given formulas in the four cases indicated. $\square$

We now extend the orthogonal decomposition of $L^2_{\mathbf{q}}G$ to any Hilbert $\mathcal{N}_{\mathbf{q}}G$-module. By Proposition 4.7, we can identify $L^2_{\mathbf{q}}G^n$ with the orthogonal sum $K^n_+ \oplus K^n_- \oplus K^n_\emptyset$.

Proposition 4.12. *Let $V \subseteq L^2_{\mathbf{q}}G^n$ be a closed subspace that is invariant with respect to the diagonal left $\mathbb{R}G$-action, and let $V_+ = \widetilde{\kappa}_+ V$, $V_- = \widetilde{\kappa}_- V$, $V_\emptyset = \widetilde{\kappa}_\emptyset V$. Then we have an orthogonal decomposition*

$$V = V_+ \oplus V_- \oplus V_\emptyset$$

with $V_+ \subseteq K^n_+$, $V_- \subseteq K^n_-$ and $V_\emptyset \subseteq K^n_\emptyset$.

Proof. By Proposition 4.8, $\widetilde{\kappa}_+$, $\widetilde{\kappa}_-$, and $\widetilde{\kappa}_\emptyset$ are all elements of $\mathcal{N}_{\mathbf{q}}G$ and define orthogonal projections from $L^2_{\mathbf{q}}G$ onto K_+, K_-, and $K_\emptyset$, respectively. It follows that diagonal left multiplication by these elements on $L^2_{\mathbf{q}}G^n$ defines orthogonal projection onto the subspaces K^n_+, K^n_-, $K^n_\emptyset$, respectively. It follows that the summands V_+, V_-, $V_\emptyset$ are orthogonal. Since V is a left $\mathcal{N}_{\mathbf{q}}G$-module, each of the summands V_+, V_-, and $V_\emptyset$ must be contained in V, so we have

$$V \supseteq V_+ \oplus V_- \oplus V_\emptyset.$$

On the other hand, since $1 = \widetilde{\kappa}_+ + \widetilde{\kappa}_- + \widetilde{\kappa}_\emptyset$, we know that $x = \widetilde{\kappa}_+ x + \widetilde{\kappa}_- x + \widetilde{\kappa}_\emptyset x$ for any $x \in V$, giving us the opposite inclusion. $\square$

To extend our decomposition of $L^2_{\mathbf{q}}G$ to a decomposition of $L^2_{\mathbf{q}}W$, we note that $L^2_{\mathbf{q}}W$ is spanned by $L^2_{\mathbf{q}}G$ and its translate $L^2_{\mathbf{q}}Gs$. By Proposition 4.2, both K_+ and K_- are also contained in $L^2_{\mathbf{q}}Gs$, suggesting the following decomposition for $L^2_{\mathbf{q}}W$.

Proposition 4.13. *We have an orthogonal decomposition of $\mathcal{N}_{\mathbf{q}}G$-modules given by*

$$L^2_{\mathbf{q}}W = K_+ \oplus K_- \oplus K_\emptyset \oplus K_\emptyset s.$$

Moreover, $K_\emptyset$ and $K_\emptyset s$ are isomorphic as $\mathcal{N}_{\mathbf{q}}G$-modules.

Proof. Right multiplication by s is a self-adjoint involution, hence an isometry. It follows that (1) $K_\emptyset$ maps isomorphically (isometrically and equivariantly with respect to the left $\mathbb{R}W$-action) to $K_\emptyset s$, and (2) preserves orthogonality in $L^2_{\mathbf{q}}W$. The latter implies that

$$L^2_{\mathbf{q}}Gs = (K_+ \oplus K_- \oplus K_\emptyset)s = (K_+ s \oplus K_- s \oplus K_\emptyset s) = (K_+ \oplus K_- \oplus K_\emptyset s),$$

where the last equality follows from Proposition 4.2. Since $L^2_{\mathbf{q}}W$ is spanned by $L^2_{\mathbf{q}}G$ and $L^2_{\mathbf{q}}Gs$, we have

$$\begin{aligned} L^2_{\mathbf{q}}W &= L^2_{\mathbf{q}}G + L^2_{\mathbf{q}}Gs \\ &= (K_+ \oplus K_- \oplus K_\emptyset) + (K_+ \oplus K_- \oplus K_\emptyset s) \\ &= K_+ \oplus K_- \oplus (K_\emptyset + K_\emptyset s). \end{aligned}$$

The only thing left to prove is that $K_\emptyset$ and $K_\emptyset s$ are orthogonal. Since G spans a dense subspace of $L^2_{\mathbf{q}}G$, we know that $\{(st)^n\widetilde{\kappa}_\emptyset \mid n \in \mathbb{Z}\}$ spans a dense subspace of $K_\emptyset$, and $\{(st)^n s\widetilde{\kappa}_\emptyset \mid n \in \mathbb{Z}\}$ spans a dense subspace of $K_\emptyset s$. It therefore suffices to prove that

$$\langle (st)^n\widetilde{\kappa}_\emptyset, (st)^m s\widetilde{\kappa}_\emptyset\rangle_{\mathbf{q}} = 0$$

for all $m, n \in \mathbb{Z}$. Using the fact that $\widetilde{\kappa}_\emptyset$ is a self adjoint idempotent and $(st)^* = (st)^{-1}$, we have

$$\langle (st)^n\widetilde{\kappa}_\emptyset, (st)^m s\widetilde{\kappa}_\emptyset\rangle_{\mathbf{q}} = \langle s(st)^{n-m}\widetilde{\kappa}_\emptyset^2, 1\rangle_{\mathbf{q}} = \langle s(st)^{n-m}\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}}. \tag{4.14}$$

But since $\widetilde{\kappa}_\emptyset$ is central, we have (for any $x \in L^2_{\mathbf{q}}W$)

$$\langle sxs\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \langle xs\widetilde{\kappa}_\emptyset, s\rangle_{\mathbf{q}} = \langle xs^2\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \langle x\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}}$$

and, similarly,

$$\langle txt\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \langle x\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}}.$$

Repeated applications of this identity then reduce (4.14) to

$$\langle (st)^n\widetilde{\kappa}_\emptyset, (st)^m s\widetilde{\kappa}_\emptyset\rangle_{\mathbf{q}} = \begin{cases} \langle s\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} & \text{if } n-m \text{ is even,} \\ \langle t\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} & \text{if } n-m \text{ is odd.} \end{cases}$$

By definition of $\widetilde{\kappa}_\emptyset$ and Proposition 4.2, we have

$$\langle s\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \langle s, 1\rangle_{\mathbf{q}} - \langle s\widetilde{\kappa}_+, 1\rangle_{\mathbf{q}} - \langle s\widetilde{\kappa}_-, 1\rangle_{\mathbf{q}} = \langle s, 1\rangle_{\mathbf{q}} - \sigma_1\langle\widetilde{\kappa}_+, 1\rangle_{\mathbf{q}} - \sigma_2\langle\widetilde{\kappa}_-, 1\rangle_{\mathbf{q}}$$

where σ_1 is $+1$ (resp., -1) if $q_sq_t < 1$ (resp. $q_sq_t > 1$) and σ_2 is $+1$ (resp., -1) if $q_s < q_t$ (resp. $q_s > q_t$). Since $s = \frac{1-q_s}{1+q_s} + \frac{2}{1+q_s}\tau_s$, we have $\langle s, 1\rangle_{\mathbf{q}} = \frac{1-q_s}{1+q_s}$, and hence by (4.10) and (4.11) we have

$$\langle s\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \frac{1-q_s}{1+q_s} - \frac{1-q_sq_t}{(1+q_s)(1+q_t)} - \frac{q_t-q_s}{(1+q_s)(1+q_t)} = 0.$$

A similar calculation gives

$$\langle t\widetilde{\kappa}_\emptyset, 1\rangle_{\mathbf{q}} = \frac{1-q_t}{1+q_t} - \frac{1-q_sq_t}{(1+q_s)(1+q_t)} - \frac{q_s-q_t}{(1+q_s)(1+q_t)} = 0.$$

This completes the proof. $\square$

We now extend our orthogonal decomposition of $L^2_{\mathbf{q}}W$ to any Hilbert $\mathcal{N}_{\mathbf{q}}W$-module. By Proposition 4.13, we can identify $L^2_{\mathbf{q}}W^n$ with $K_+^n \oplus K_-^n \oplus (K_\emptyset \oplus K_\emptyset s)^n$.

Proposition 4.15. *Let $V \subseteq L^2_{\mathbf{q}}W^n$ be a closed subspace that is invariant with respect to the diagonal left $\mathbb{R}W$-action, and let $V_+ = \widetilde{\kappa}_+V$, $V_- = \widetilde{\kappa}_-V$, $V_\emptyset = \widetilde{\kappa}_\emptyset V$. Then we have an orthogonal decomposition*

$$V = V_+ \oplus V_- \oplus V_\emptyset$$

with $V_+ \subseteq K_+^n$, $V_- \subseteq K_-^n$ *and* $V_\emptyset \subseteq (K_\emptyset \oplus K_\emptyset s)^n$.

Proof. The proof is the same as the proof of Proposition 4.12. The only difference is that as an operator on $L^2_{\mathbf{q}}W$, the idempotent $\widetilde{\kappa}_\emptyset$ projects onto the orthogonal complement of $K_+ \oplus K_-$ *in* $L^2_{\mathbf{q}}W$, which is now $K_\emptyset \oplus K_\emptyset s$. □

Any $\mathcal{N}_{\mathbf{q}}W$-module is naturally an $\mathcal{N}_{\mathbf{q}}G$-module, hence we can ask for its von Neumann dimension with respect to either structure. The following lemma relates the two.

Lemma 4.16. *Let* $V \subseteq L^2_{\mathbf{q}}W^n$ *be a Hilbert* $\mathcal{N}_{\mathbf{q}}W$*-module. Then*

(1) $\dim_W^{\mathbf{q}} V_+ = \dim_G^{\mathbf{q}} V_+$,
(2) $\dim_W^{\mathbf{q}} V_- = \dim_G^{\mathbf{q}} V_-$, *and*
(3) $\dim_W^{\mathbf{q}} V_\emptyset = \frac{1}{2}\dim_G^{\mathbf{q}} V_\emptyset$.

Proof. We identify $L^2_{\mathbf{q}}W^n$ with $K_+^n \oplus K_-^n \oplus (K_\emptyset \oplus K_\emptyset s)^n$. To prove (1) and (2), let $\pi_+ : K_+^n \to K_+^n$ and $\pi_- : K_-^n \to K_-^n$ denote orthogonal projections onto V_+ and V_-, respectively. By composing projections, we then have that the orthogonal projection from $L^2_{\mathbf{q}}W^n$ to V_+, and hence from $L^2_{\mathbf{q}}G^n$ to $V+$, are both given by $\pi_+\widetilde{\kappa}_+$. Similarly, the orthogonal projection from $L^2_{\mathbf{q}}W^n$ to V_- is given by $\pi_-\widetilde{\kappa}_-$. Let $\epsilon_1, \dots, \epsilon_n$ be the standard basis for $L^2_{\mathbf{q}}W^n$ as a free $\mathcal{N}_{\mathbf{q}}W$-module. Then it can also be regarded as the standard basis for the subspace $L^2_{\mathbf{q}}G^n$ regarded as a free $\mathcal{N}_{\mathbf{q}}G$-module. Hence, we have

$$\dim_G^{\mathbf{q}} V_+ = \sum_{i=1}^n \langle \pi_+(\widetilde{\kappa}_+\epsilon_i), \epsilon_i\rangle = \dim_W^{\mathbf{q}} V_+,$$

and

$$\dim_G^{\mathbf{q}} V_- = \sum_{i=1}^n \langle \pi_-(\widetilde{\kappa}_-\epsilon_i), \epsilon_i\rangle = \dim_W^{\mathbf{q}} V_-.$$

To prove (3), we let $\pi_\emptyset : (K_\emptyset \oplus K_\emptyset s)^n \to (K_\emptyset \oplus K_\emptyset s)^n$ be orthogonal projection onto $V_\emptyset$. Again by composing projections, we have that the orthogonal projection from $L^2_{\mathbf{q}}W^n$ to $V_\emptyset$ is given by $\pi_\emptyset k_\emptyset$, and hence

$$\dim_W^{\mathbf{q}} V_\emptyset = \sum_{i=1}^n \langle \pi_\emptyset(k_\emptyset\epsilon_i), \epsilon_i\rangle.$$

To calculate the dimension of $V_\emptyset$ as an $\mathcal{N}_{\mathbf{q}}G$-module, we shall embed it in the free $\mathcal{N}_{\mathbf{q}}G$-module $L^2_{\mathbf{q}}G^n \oplus L^2_{\mathbf{q}}G^n$. We let $\epsilon_1, \dots, \epsilon_n$ denote the standard basis for the first summand of $L^2_{\mathbf{q}}G^n \oplus L^2_{\mathbf{q}}G^n$ and $\epsilon'_1, \dots, \epsilon'_n$ denote the standard basis for the second summand. We then define

$$\phi : (K_\emptyset \oplus K_\emptyset s)^n \to L^2_{\mathbf{q}}G^n \oplus L^2_{\mathbf{q}}G^n$$

by $\phi(x_1 + x'_1 s, \dots, x_n + x'_n s) \mapsto ((x_1, \dots, x_n), (x'_1, \dots, x'_n))$. This map is an isometric embedding, equivariant with respect to the left $\mathbb{R}G$-action, and the image is $K_\emptyset^n \oplus K_\emptyset^n$. As an $\mathcal{N}_{\mathbf{q}}G$-module $(K_\emptyset \oplus K_\emptyset s)^n$ is generated by $\widetilde{\kappa}_\emptyset\epsilon_1, \dots, \widetilde{\kappa}_\emptyset\epsilon_n$ and $\widetilde{\kappa}_\emptyset s\epsilon_1, \dots, \widetilde{\kappa}_\emptyset s\epsilon_n$. The images of these generators are given by $\phi(\widetilde{\kappa}_\emptyset\epsilon_i) = \widetilde{\kappa}_\emptyset\epsilon_i$ and $\phi(\widetilde{\kappa}_\emptyset s\epsilon_i) = \widetilde{\kappa}_\emptyset\epsilon'_i$. As an $\mathcal{N}_{\mathbf{q}}G$-module $V_\emptyset$ is isomorphic to the image $\phi(V_\emptyset) \subseteq L^2_{\mathbf{q}}G^n \oplus L^2_{\mathbf{q}}G^n$, and orthogonal

projection onto this image is given by the composition $\phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset$. We can therefore compute

$$
\begin{aligned}
\dim_G^{\mathbf{q}} V_\emptyset &= \dim_G^{\mathbf{q}} \phi(V_\emptyset) \\
&= \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i'), \epsilon_i'\rangle && (\text{definition of } \dim_G^{\mathbf{q}}) \\
&= \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset^2(\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset^2(\epsilon_i'), \epsilon_i'\rangle && (\widetilde{\kappa}_\emptyset \text{ is idempotent}) \\
&= \sum_{i=1}^{n} \langle \widetilde{\kappa}_\emptyset\phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle \widetilde{\kappa}_\emptyset\phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i'), \epsilon_i'\rangle \\
&\qquad (\phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset \text{ is } \mathcal{N}_{\mathbf{q}}G\text{-equivariant}) \\
&= \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i), \widetilde{\kappa}_\emptyset\epsilon_i\rangle + \sum_{i=1}^{n} \langle \phi\pi_\emptyset\phi^{-1}\widetilde{\kappa}_\emptyset(\epsilon_i'), \widetilde{\kappa}_\emptyset\epsilon_i'\rangle && (\widetilde{\kappa}_\emptyset \text{ is self-adjoint}) \\
&= \sum_{i=1}^{n} \langle \phi\pi_\emptyset(\widetilde{\kappa}_\emptyset\epsilon_i), \phi(\widetilde{\kappa}_\emptyset\epsilon_i)\rangle + \sum_{i=1}^{n} \langle \phi\pi_\emptyset(\widetilde{\kappa}_\emptyset s\epsilon_i), \phi(\widetilde{\kappa}_\emptyset s\epsilon_i\rangle && (\text{definition of } \phi) \\
&= \sum_{i=1}^{n} \langle \pi_\emptyset(\widetilde{\kappa}_\emptyset\epsilon_i), \widetilde{\kappa}_\emptyset\epsilon_i\rangle + \sum_{i=1}^{n} \langle \pi_\emptyset(\widetilde{\kappa}_\emptyset s\epsilon_i), \widetilde{\kappa}_\emptyset s\epsilon_i\rangle && (\phi \text{ is an isometry}) \\
&= \sum_{i=1}^{n} \langle \widetilde{\kappa}_\emptyset\pi_\emptyset(\widetilde{\kappa}_\emptyset\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle s\widetilde{\kappa}_\emptyset\pi_\emptyset(\widetilde{\kappa}_\emptyset s\epsilon_i), \epsilon_i\rangle \\
&\qquad (s \text{ and } \widetilde{\kappa}_\emptyset \text{ are self-adjoint}) \\
&= \sum_{i=1}^{n} \langle \pi_\emptyset(\widetilde{\kappa}_\emptyset^2\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle \pi_\emptyset(s\widetilde{\kappa}_\emptyset^2 s\epsilon_i), \epsilon_i\rangle && (\pi_\emptyset \text{ is } \mathcal{N}_{\mathbf{q}}W\text{-equivariant}) \\
&= \sum_{i=1}^{n} \langle \pi_\emptyset(\widetilde{\kappa}_\emptyset\epsilon_i), \epsilon_i\rangle + \sum_{i=1}^{n} \langle \pi_\emptyset(\widetilde{\kappa}_\emptyset\epsilon_i), \epsilon_i\rangle \\
&\qquad (\widetilde{\kappa}_\emptyset \text{ is a central idempotent and } s^2 = 1) \\
&= 2\dim_W^{\mathbf{q}} V_\emptyset.
\end{aligned}
$$

□

5. Kernels of $\mathbb{R}G$ and $\mathbb{R}W$-matrices

In this section, we consider only those $\mathcal{N}_{\mathbf{q}}G$-modules (respectively, $\mathcal{N}_{\mathbf{q}}W$-modules) that are given by kernels of right multiplication by $\mathbb{R}G$-matrices (resp., $\mathbb{R}W$-matrices). The fundamental fact that our arguments rely on is that the submodules $K_+, K_-, K_\emptyset \subseteq L^2_{\mathbf{q}}G$ are irreducible in the sense that right multiplication by an element of $\mathbb{R}G$ is either the zero map or an isomorphism. For K_+ and K_- this is obvious since they are each

spanned by a single vector, but for $K_\emptyset$ we need the fact that there are no other st-eigenvectors in $L^2_{\mathbf{q}}G$.

Proposition 5.1. *For any element $y \in \mathbb{R}G$, let $R_y : K_\emptyset \to K_\emptyset$ denote (right) multiplication by y. Then*

$$\ker R_y = \begin{cases} K_\emptyset & \text{if } y = 0, \\ 0 & \text{if } y \neq 0. \end{cases}$$

Proof. Since G is infinite cyclic generated by st, y is a Laurent polynomial in st, hence can be factored as

$$y = C \cdot (st)^{-n} \cdot p(st)$$

where n is an integer, C is a nonzero real constant, and $p(z)$ is a polynomial in z with real coefficients. Factoring this polynomial gives

$$y = C\ (st)^{-n}\ (st - \lambda_1) \cdots (st - \lambda_k),$$

where the $\lambda_i \in \mathbb{C}$ are the roots of $p(z)$. If $R_y(x) = 0$ for some nonzero $x \in K_\emptyset$, then at least one of the linear factors $(st - \lambda_i)$ must have nontrivial kernel, contradicting Theorem 3.21. □

Now we suppose M is an $(m \times n)$-matrix with $\mathbb{R}G$-entries. We let $R_M : L^2_{\mathbf{q}}G^m \to L^2_{\mathbf{q}}G^n$ denote right multiplication by M. Then $\ker R_M$ is a left $\mathcal{N}_{\mathbf{q}}G$-module, hence, by Proposition 4.12, decomposes as

$$\ker R_M = (\ker R_M)_+ \oplus (\ker R_M)_- \oplus (\ker R_M)_\emptyset.$$

Moreover, each summand can be regarded as the kernel of right multiplication by M on the corresponding invariant subspace of $L^2_{\mathbf{q}}G^m = K_+^m \oplus K_-^m \oplus K_\emptyset^m$. More precisely, if $R_M^+ : K_+^m \to K_+^m$, $R_M^- : K_-^m \to K_-^m$, and $R_M^\emptyset : K_\emptyset^m \to K_\emptyset^m$ each denotes right multiplication by the matrix M, then

$$(\ker R_M)_+ = \ker R_M^+, \qquad (\ker R_M)_- = \ker R_M^-, \qquad \text{and } \ (\ker R_M)_\emptyset = \ker R_M^\emptyset.$$

Lemma 5.2. *Let M be a matrix with $\mathbb{R}G$-entries, and let R_M^+, R_M^-, and $R_M^\emptyset$ denote right multiplication by M on K_+^m, K_-^m, and $K_\emptyset^m$, respectively. Then there exist $\mathcal{N}_{\mathbf{q}}G$-module isomorphisms*

$$\ker R_M^+ \cong K_+^a, \quad \ker R_M^- \cong K_-^b, \quad \textit{and} \ \ \ker R_M^\emptyset \cong K_\emptyset^c,$$

for some choice of integers $a, b, c \in \{0, 1, \ldots, m\}$.

Proof. Adding a zero column to M does not effect the kernel of R_M^+, R_M^-, or $R_M^\emptyset$, and adding a zero row only alters the kernel by a free summand of K_+, K_-, or $K_\emptyset$, respectively. We can therefore assume that M is a square matrix of size $m \times m$. The entries of M are elements of $\mathbb{R}G$, which we regard as the ring of Laurent polynomials in $z = st$ over $\mathbb{R}$. Since right multiplication by $z = st$ (a unitary operator on $L^2_{\mathbf{q}}G^n$) defines an $\mathcal{N}_{\mathbf{q}}G$-module automorphism of K_+^m, K_-^m, and $K_\emptyset^m$, resp., we can multiply M by any power of z without changing the kernel of R_M^+, R_M^-, or $R_M^\emptyset$, resp. Thus, we can assume that M has polynomial entries. Since polynomials over $\mathbb{R}$ form a principal ideal domain, we can multiply M on the right and left by invertible matrices (over $\mathbb{R}G$) to

obtain a diagonal matrix. Hence the proof of the lemma reduces to the case where M is a diagonal matrix $\mathrm{diag}(y_1, \dots, y_m)$. Finally we simply recall, from Proposition 5.1 and the paragraph preceding it, that right multiplication on K_+, K_-, or $K_\emptyset$ by any element $y_i \in \mathbb{R}G$ is either an isomorphism or the zero map. The result follows. □

Finally, we consider $\mathcal{N}_{\mathbf{q}}W$-modules that are kernels of $\mathbb{R}W$-matrices. Let M be an $(m \times n)$-matrix with $\mathbb{R}W$-entries, and let $R_M : L^2_{\mathbf{q}}W^m \to L^2_{\mathbf{q}}W^n$ denote right multiplication by M. As in the case of $\mathbb{R}G$-matrices, we obtain a decomposition of left $\mathcal{N}_{\mathbf{q}}W$-modules:

$$\ker R_M = \ker R_M^+ \oplus \ker R_M^- \oplus \ker R_M^\emptyset, \tag{5.3}$$

where $R_M^+ : K_+^m \to K_+^m$, $R_M^- : K_-^m \to K_-^m$, and $R_M^\emptyset : (K_\emptyset \oplus K_\emptyset s)^m \to (K_\emptyset \oplus K_\emptyset s)^m$ each denotes right multiplication by the matrix M. These three summands are also left $\mathcal{N}_{\mathbf{q}}G$-modules, however, in order to use Lemma 5.2 , we need to know that as $\mathcal{N}_{\mathbf{q}}G$-modules they are isomorphic to kernels of $\mathbb{R}G$-matrices.

Lemma 5.4. *Let M be an $(m \times n)$-matrix with entries in $\mathbb{R}W$. Then there exist $(m \times n)$-matrices M_+ and M_-, and a $(2m \times 2n)$-matrix $M_\emptyset$ all with entries in $\mathbb{R}G$ such that as $\mathcal{N}_{\mathbf{q}}G$-modules,*

$$\ker R_M^+ \cong \ker R_{M_+}, \quad \ker R_M^- \cong \ker R_{M_-}, \quad \textit{and} \quad \ker R_M^\emptyset \cong \ker R_{M_\emptyset},$$

where R_{M_+} denotes right-multiplication by M_+ on K_+^m, R_{M_-} denotes right-multiplication by M_- on K_-^m, and $R_{M_\emptyset}$ denotes right-multiplication by $M_\emptyset$ on $K_\emptyset^{2m}$.

Proof. Any element y in $\mathbb{R}W$ can be written in the form $y = y_1(z) + y_2(z)s$ where $y_1(z)$ and $y_2(z)$ are Laurent polynomials in $z = st$. Moreover, since $(st)^n s = s(ts)^n = s(st)^{-n}$, any Laurent polynomial $f(z) \in \mathbb{R}G$ satisfies the relation $f(z)s = sf(z^{-1})$ in $\mathbb{R}W$. These same properties hold for any matrix M with $\mathbb{R}W$ entries. Given such a matrix M, we let $M = M_1(z) + M_2(z)s$ where $M_(z)$ and $M_2(z)$ are $(m \times n)$-matrices with entries in $\mathbb{R}G$. Given $x \in K_+^m$, we have $x = x\widetilde{\kappa}_+$, so

$$\begin{aligned}
xM &= x\widetilde{\kappa}_+(M_1(z) + M_2(z)s) \\
&= x\widetilde{\kappa}_+M_1(z) + x\widetilde{\kappa}_+sM_2(z^{-1}) \\
&= x\widetilde{\kappa}_+M_1(z) \pm x\widetilde{\kappa}_+M_2(z^{-1}) \qquad \text{(sign depending on } \mathbf{q}) \\
&= x\widetilde{\kappa}_+(M_1(z) \pm M_2(z^{-1})) \\
&= x(M_1(z) \pm M_2(z^{-1})).
\end{aligned}$$

In other words, right multiplication by M on K_+^m is the same as right multiplication by $M_1(z) \pm M_2(z^{-1}))$, which has entries in $\mathbb{R}G$. Letting M_+ be the matrix $M_+ = M_1(z) \pm M_2(z^{-1}))$, we therefore have $\ker R_M^+ \cong \ker R_{M_+}$, as desired. A similar argument works for R_M^- acting on K_-^m.

For $x \in (K_\emptyset \oplus K_\emptyset s)^m$, we express it as $x = x_1 + x_2 s$ where $x_1, x_2 \in K_\emptyset^m$. Then

$$\begin{aligned} xM &= (x_1 + x_2 s)(M_1(z) + M_2(z)s) \\ &= x_1(M_1(z) + M_2(z)s) + x_2 s(M_1(z) + M_2(z)s) \\ &= x_1 M_1(z) + x_1 M_2(z)s + x_2 M_1(z^{-1})s + x_2 M_2(z^{-1}) \\ &= [x_1 M_1(z) + x_2 M_2(z^{-1})] + [x_1 M_2(z) + x_2 M_1(z^{-1})]s. \end{aligned}$$

It follows that if we identify $(K_\emptyset \oplus K_\emptyset s)^m$ with $K_\emptyset^m \oplus K_\emptyset^m$ (using the $\mathcal{N}_{\mathbf{q}}G$-isomorphism $x_1 + x_2 s \mapsto (x_1, x_2)$), then right multiplication by M corresponds to right multiplication by the $(2m \times 2n)$ block matrix

$$M_\emptyset = \begin{bmatrix} M_1(z) & M_2(z) \\ M_2(z^{-1}) & M_1(z^{-1}) \end{bmatrix}.$$

Hence the two matrices M and $M_\emptyset$ will have isomorphic kernels (as $\mathcal{N}_{\mathbf{q}}G$-modules). $\square$

As in the introduction, we define $\Lambda_{\mathbf{q}}$ to be the additive subgroup of $\mathbb{R}$ generated by $\frac{1}{1+q_s}$, $\frac{1}{1+q_t}$, and 1.

Proposition 5.5. *The subgroup $\Lambda_{\mathbf{q}}$ coincides with the subgroup generated by $\dim_G^{\mathbf{q}} K_+$, $\dim_G^{\mathbf{q}} K_-$, and $\dim_G^{\mathbf{q}} L^2_{\mathbf{q}}G$.*

Proof. Let $\Lambda'_{\mathbf{q}}$ denote the subgroup of $\mathbb{R}$ generated by $\dim_G^{\mathbf{q}} K_+$, $\dim_G^{\mathbf{q}} K_-$, and $\dim_G^{\mathbf{q}} L^2_{\mathbf{q}}G$ $(= 1)$. By Lemma 4.9, $\Lambda'_{\mathbf{q}}$ is the subgroup generated by

$$a = \frac{1 - q_s q_t}{(1+q_s)(1+q_t)}, \qquad b = \frac{q_s - q_t}{(1+q_s)(1+q_t)},$$

and 1. The equations

$$\begin{aligned} a &= \frac{1}{1+q_s} + \frac{1}{1+q_t} - 1, \text{and} \\ b &= \frac{1}{1+q_t} - \frac{1}{1+q_s} \end{aligned}$$

show that $\Lambda'_{\mathbf{q}} \subseteq \Lambda_{\mathbf{q}}$. Since these equations can be solved for $\frac{1}{1+q_s}$ and $\frac{1}{1+q_t}$ in terms of a, b, and 1, we have the reverse inclusion as well. $\square$

We now prove the main theorem of the paper.

Theorem 5.6. *If M is any $(m \times n)$-matrix with entries in $\mathbb{R}W$ and $R_M : L^2_{\mathbf{q}}W^m \to L^2_{\mathbf{q}}W^n$ denotes right multiplication by M, then*

$$\dim_W^{\mathbf{q}} \ker R_M \in \Lambda_{\mathbf{q}}.$$

Moreover, $\Lambda_{\mathbf{q}}$ is the smallest subgroup of $\mathbb{R}$ having this property.

Proof. By (5.3), we have

$$\dim_W^{\mathbf{q}} \ker R_M = \dim_W^{\mathbf{q}} \ker R_M^+ + \dim_W^{\mathbf{q}} \ker R_M^- + \dim_W^{\mathbf{q}} \ker R_M^\emptyset,$$

hence by Lemma 4.16, we have

$$\dim_W^{\mathbf{q}} \ker R_M = \dim_G^{\mathbf{q}} \ker R_M^+ + \dim_G^{\mathbf{q}} \ker R_M^- + \frac{1}{2} \dim_G^{\mathbf{q}} \ker R_M^{\emptyset}. \tag{5.7}$$

By Lemma 5.4, all of these $\mathcal{N}_{\mathbf{q}}G$-modules are isomorphic to kernels of $\mathbb{R}G$-matrices, hence by Lemma 5.2, we have

$$\begin{aligned}
\dim_G^{\mathbf{q}} \ker R_M^+ &= \dim_G^{\mathbf{q}} K_+^a, \\
\dim_G^{\mathbf{q}} \ker R_M^- &= \dim_G^{\mathbf{q}} K_-^b, \\
\dim_G^{\mathbf{q}} \ker R_M^{\emptyset} &= \dim_G^{\mathbf{q}} K_{\emptyset}^c
\end{aligned} \tag{5.8}$$

for some integers a, b, c. It then follows from (5.7) and Lemma 4.9 that $\dim_W^{\mathbf{q}} R_M$ is contained in the subgroup generated by $\dim_G^{\mathbf{q}} K_{\pm}$, $\frac{1}{1+q_s}$, $\frac{1}{1+q_t}$, $\frac{q_s}{1+q_s}$ $(= 1 - \frac{1}{1+q_s})$, and $\frac{q_t}{1+q_t}$ $(= 1 - \frac{1}{1+q_t})$. By Proposition 5.5, this subgroup is $\Lambda_{\mathbf{q}}$.

It remains to show that no smaller subgroup than $\Lambda_{\mathbf{q}}$ will contain all possible dimensions of kernels of $\mathbb{R}W$-matrices. This follows from the fact that K_+, K_-, and $L^2_{\mathbf{q}}W$ can all be realized as kernels of right multiplication by $1 - st$, $1 + st$, and 0 (respectively). By Proposition 5.5, the dimensions of these $\mathcal{N}_{\mathbf{q}}W$-modules generate $\Lambda_{\mathbf{q}}$ $\square$

As a final observation, we note that Proposition 5.1 and Lemmas 5.2 and 5.4 are independent of the parameters $\mathbf{q}$. Hence, for a fixed $\mathbb{R}W$-matrix M, the equations (5.7) and (5.8) combine to give

$$\dim_W^{\mathbf{q}} \ker R_M = a \cdot \dim_G^{\mathbf{q}} K_+ + b \cdot \dim_G^{\mathbf{q}} K_- + \frac{c}{2} \cdot \dim_G^{\mathbf{q}} K_{\emptyset}$$

where the integers a, b, c are constant with respect to $\mathbf{q}$. Applying Lemma 4.9, we obtain the following corollary.

Corollary 5.9. *For a fixed $\mathbb{R}W$-matrix M, the dimension $\dim_W^{\mathbf{q}} \ker R_M$ is a continuous, piecewise-rational function of q_s and q_t, with breaks only along the curves $q_s = q_t$ and $q_s q_t = 1$.*

References

1. Tim Austin, *Rational group ring elements with kernels having irrational dimension*, arXiv:0909.2360v2 (2009).
2. Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, vol. 32, Princeton University Press, Princeton, NJ, 2008. MR MR2360474
3. Michael W. Davis, Jan Dymara, Tadeusz Januszkiewicz, and Boris Okun, *Weighted L^2-cohomology of Coxeter groups*, Geom. Topol. **11** (2007), 47–138. MR MR2287919 (2008g:20084)
4. Jacques Dixmier, *Les C^*-algèbres et leurs représentations*, Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics], Éditions Jacques Gabay, Paris, 1996, Reprint of the second (1969) edition. MR MR1452364 (98a:46066)
5. Jan Dymara, *Thin buildings*, Geom. Topol. **10** (2006), 667–694. MR 2240901 (2007h:20027)
6. Łukasz Grabowski, *On the Atiyah problem for the lamplighter groups*, arXiv:1009.0229 (2010).
7. Peter Linnell, Boris Okun, and Thomas Schick, *The strong Atiyah conjecture for right-angled Artin and Coxeter groups*, Geometriae Dedicata (to appear).

8. Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR 1926649 (2003m:58033)
9. Boris Okun and Richard Scott, *L^2-homology and reciprocity for right-angled coxeter groups*, Fund. Math. **214** (2011), no. 1, 27–56.
10. Mikaël Pichot, Thomas Schick, and Andrzej Zuk, *Closed manifolds with transcendental L2-betti numbers*, arXiv:1005.1147 (2010).
11. Louis Solomon, *A decomposition of the group algebra of a finite Coxeter group*, J. Algebra **9** (1968), 220–239. MR 0232868 (38 #1191)

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